Morse Homology and Problems of Prescribed Mean Curvature.

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Counting Minimal Surfaces - I

Theorem, Simon, Smith (1982)

For every riemannian metric g over $M := S^3$ there exists at least 1 embedded minimal sphere.

Theorem, Grüter, Jost (1986)

For every riemannian metric g over $M := B^3$ such that ∂M is locally strictly convex there exists at least 1 embedded minimal disk meeting ∂M orthogonally.

Counting Minimal Surfaces - II

Theorem, White (1989)

For every riemannian metric g over $M := S^3$ such that $\text{Ric}_g > 0$,there exists at least 1 embedded minimal torus.

Theorem, Máximo, Nunes, S. (2013)

For every riemannian metric g over $M := B^3$ such that $\text{Ric}_g \geq 0$ and ∂M is locally strictly convex there exists at least 1 embedded minimal annulus meeting ∂M orthogonally.

Counting Minimal Surfaces - III

Theorem (Incomplete), Jost (1989)

For every riemannian metric g over $M := S^3$ there exist at least 4 embedded minimal 2-spheres.

Theorem (Incomplete), Jost (1989)

For every riemannian metric g over $M := B^3$ such that ∂M has positive mean curvature there exist at least 3 embedded minimal disks meeting ∂M orthogonally.

Counting Minimal Surfaces - IV

Conjecture, White

For generic riemannian metrics g over $M := S^3$ such that $\mathrm{Ric}_g > 0$ there exist at least 9 embedded minimal tori.

Conjecture, White for Balls

For generic riemannian metrics g over $M := B^3$ such that $\mathrm{Ric}_g \geq 0$ and ∂M is locally strictly convex there exist at least 3 embedded minimal annuli.

The space of embeddings

Minimal surfaces are critical points of the area functional.

Lower bounds for the number of minimal surfaces are lower bounds for the number of critical points of the area functional.

This is a **topological invariant** of the space of embeddings.

It follows by Morse theory that if the space of embeddings were finite dimensional, a lower bound would be given by the sum of its Betti numbers.

Numerology of embedded spheres

Heuristically, the space of embedded spheres in S^3 has the same topology as the space of embedded, **minimal** spheres in S^3.

Embedded minimal spheres in S^3 are all equatorial spheres.

These are parametrised, modulo reflection, by their centres.

The space of embedded minimal spheres in S^3 is homeomorphic to $\mathbb{RP}^3$.

The sum of the Betti numbers of $\mathbb{RP}^3$ is 4.

Numerology of embedded disks

Heuristically, the space of embedded disks in B^3 meeting ∂B^3 orthogonally has the same topology as the space of embedded, **minimal** disks in B^3 having the same property.

Embedded minimal disks in B^3 having this property are all equatorial planes.

These are parametrised by their axes.

The space of embedded minimal spheres in B^3 meeting ∂B^3 orthogonally is homeomorphic to $\mathbb{RP}^2$.

The sum of the Betti numbers of $\mathbb{RP}^2$ is 3.

Numerology of embedded tori

Heuristically, the space of embedded tori in S^3 has the same topology as the space of embedded, **minimal** tori in S^3.

Theorem, Brendle (2012)

The only embedded minimal tori in S^3 are the Clifford tori.

These are parametrised by $\mathbb{RP}^2 \times \mathbb{RP}^2$.

The sum of the Betti numbers of $\mathbb{RP}^2 \times \mathbb{RP}^2$ is 9.

Numerology of embedded annuli

Heuristically, the space of embedded annuli in B^3 meeting ∂B^3 orthogonally has the same topology as the space of embedded, **minimal** annuli in B^3 having the same property.

Conjecture

Embedded minimal annuli in B^3 having this property are all rotationally symmetric.

Rotationally symmetric annuli are parametrised by their axes.

The space of embedded minimal annuli in B^3 meeting ∂B^3 orthogonally is homeomorphic to $\mathbb{RP}^2$.

The sum of the Betti numbers of $\mathbb{RP}^2$ is 3.

Morse homology

Let X be a compact **finite dimensional** manifold.

Let $f : X \to \mathbb{R}$ be a (generic) smooth function.

For all k, let $\text{Crit}(f; k)$ be the set of critical points of f of Morse index k.

For all k, let C_k be the $\mathbb{Z}_2$ module

$$C_k := \{\phi : \text{Crit}(f, k) \to \mathbb{Z}_2\}.$$

Define $\partial_k : C_k \to C_{k-1}$ such that $\langle \partial_k \delta_X, \delta_Y \rangle$ equals the number of gradient flows modulo 2 from X to Y.

$$\partial_{k-1} \circ \partial_k = 0.$$

The homology H_k of (C_k, ∂_k) is the **Morse homology** of (X, f).

Compactness results

The spaces of embedded surfaces in $\mathbb{S}^3$ and B^3 is **infinite dimensional**.

This is compensated for using compactness results.

Theorem, Choi, Schoen (1985)

Given a Riemannian metric g over $M := S^3$ such that $\mathrm{Ric}_g > 0$, the space of embedded surfaces in M that are minimal with respect to g is compact.

Theorem, Fraser, Li (2015)

Given a Riemannian metric g over $M := B^3$, such that $\mathrm{Ric}_g \geq 0$ and ∂M is convex, the space of embedded surfaces in M that are minimal with respect to g and which meet ∂M orthogonally is compact.

Mean curvature flows

A **mean curvature flow** is a map $e : [0, T[\times\Sigma \to M$ such that, for all t, $e_t := e(t, \cdot)$ is an embedding, and

$$\langle \partial_t e_t, N_t \rangle = -H_t.$$

Mean curvature flows are L^2-gradient flows of the area functional.

Any form of Morse theory for the space of **smooth** immersions requires a good understanding of the long-time properties of mean curvature flows.

Eternal mean curvature flows

An **eternal mean curvature flow** is a mean curvature flow defined for all time.

Morse homology theory really only depends upon the following result.

Conjecture

Given a Riemannian metric g over $M := S^3$ such that $\mathrm{Ric}_g > 0$, the space of complete mean curvature flows in (M, g) of uniformly bounded area is compact in the C^∞_{loc}-topology.

Eternal forced mean curvature flows

Given a smooth function $f : M \to [0,\infty[$ a **forced mean curvature flow** is a map $e : [0,T[\times\Sigma \to M$ such that, for all t, $e_t := e(t,\cdot)$ is an embedding, and

$$\langle \partial_t e_t, N_t\rangle = f\circ e_t - H_t.$$

Theorem, S. (2015)

Let $M := T^{d+1}$ be a $(d+1)$-dimensional torus. For all $f : M \to]0,\infty[$ such that

$$\sup_{\xi_x\in TM,\|\xi_{\|}=1}\|D^2f(x)(\xi_x,\xi_x)\| < (3-2\sqrt{2})\inf_{x\in M}|f(x)|^3\,,$$

the space of eternal forced mean curvature flows with forcing term f is compact in the C^∞_{loc}-sense (subject to technical hypotheses...)

The "area-minus-volume" functional

Let $f : T^{d+1} \to]0, \infty[$ be a smooth, positive function.

Let $e : S^d \to T^{d+1}$ be a locally strictly convex immersion.

The immersion e lifts to an **embedding** $\tilde{e} : S^d \to \mathbb{R}^{d+1}$.

The "area-minus-volume" functional is given by

$$\mathcal{F}(e) := \text{Area}(e) - \int_{\text{Int}(e)} f(x) \text{dVol}_x.$$

The forced mean curvature flow is the L^2-gradient flow of the "area-minus-volume" functional.

So... what we can prove.

Critical points of the "area-minus-volume" functional are hypersurfaces of mean curvature prescribed by f.

Theorem (in preparation)

If $d \geq 2$ and if T^{d+1} is a $(d+1)$-dimensional torus, then, for generic, smooth functions $f : T^{d+1} \to]0, \infty[$ such that

$$\mathrm{sup}_{x \in T^{d+1}, \|\xi\|=1} \|D^2 f(\xi, \xi)\| < (3 - 2\sqrt{2}) \mathrm{inf}_{x \in T^{d+1}} |f(x)|,$$

there exist at least 2^{d+1} distinct Alexandrov embeddings $e : S^d \to T^{d+1}$ of mean curvature prescribed at every point by f.

Numerology of embedded spheres

Heuristically, the space of embedded spheres in T^{d+1} has the same topology as the space of embedded, **CMC** spheres in T^{d+1}.

CMC spheres are parametrised by their centres and their curvatures, that is $T^{d+1}\times]0,\infty[$.

The sum of the Betti numbers is therefore 2^{d+1}.

Understanding the compactness result - I

Consider a real number $C \geq 1$.

We say that a locally strictly convex embedding $e : S^d \to \mathbb{R}^{d+1}$ is C-pinched whenever its principal curvatures satisfy at every point

$$\kappa_d(x) < C\kappa_1(x).$$

We say that a locally strictly convex embedded $e : S^d \to \mathbb{R}^{d+1}$ is C-non-collapsed whenever, for all $x \in S^d$, the hypersphere of curvature $C\kappa_1(x)$ which is an interior tangent to $\mathrm{Im}(e)$ at $e(x)$ is contained entirely within $\mathrm{Im}(e)$.

Understanding the compactness result - II

Consider the function

$$\phi(t) := \frac{(t-1)}{t(t+1)}.$$

This function has 2 well-defined inverses

$$\lambda :]0, (3-2\sqrt{2})[\to]1, (1+\sqrt{2})[$$
$$\Lambda :](3-2\sqrt{2}), +\infty] \to]1, (1+\sqrt{2})[$$

Understanding the compactness result - III

Given $f : T^{d+1} \to]0,\infty[$, define

$$\lambda_f = \lambda\left(\sup_{\xi_x \in T^{d+1}, \|\xi_x\|=1} D^2 f(\xi_x,\xi_x) / \inf_{x\in T^{d+1}} f(x)^3\right)$$

$$\Lambda_f = \Lambda\left(\sup_{\xi_x \in T^{d+1}, \|\xi_x\|=1} D^2 f(\xi_x,\xi_x) / \inf_{x\in T^{d+1}} f(x)^3\right)$$

Theorem, S. (2016)

If $e_t : \mathbb{R} \times S^d \to T^{d+1}$ is an eternal forced mean curvature flow of bounded type with sub-critical forcing term f, and if e_t is pointwise λ_f-pinched and λ_f non-collapsed, then e_t is pointwise Λ_f-pinched and Λ_f-non-collapsed.

谢谢