On the asymptotic geometry of finite-type *k*-surfaces in 3-dimensional hyperbolic space

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Definitions

$\mathbb{H}^3$ is 3-dimensional hyperbolic space.

A k-**surface** in a pair (S, e), where S is a smooth surface and $e : S \to \mathbb{H}^3$ is an immersion of constant extrinsic curvature equal to k.

A k-surface has **finite type** whenever the induced metric is complete and has finite area.

Examples - Fermi coordinates

The Fermi metric of $\mathbb{H}^2$ about a complete geodesic is

$$g = dr^2 + \cosh(r)^2 dt^2.$$

If $\gamma := (r, t) : \mathbb{R} \to \mathbb{R}^2$ satisfis

$$r(s) = \operatorname{arcsinh}(A\cosh(\sqrt{1-k}s) + B\sinh(\sqrt{1-k}s)) \text{ and}$$
$$\dot{h}(s)^2 = \frac{1}{\cosh(r(s))^2}(1 - \dot{r}(s)^2),$$

then the surface of revolution $\Sigma_k(A, B)$ generated by rotating γ about the t-axis is a k-surface.

Examples - Types of surfaces of revolution

$\lim_{B \to 0} \Sigma_k(0, B)$ is the unit normal circle bundle over a complete geodesic.

This models degenerate limits of sequences of k-surfaces.

When $A > 0$, $\Sigma_k(-A, -A)$ has a finite area cusp at $+\infty$.

This models the geometry at infinity of finite-type k-surfaces.

k-ends and abstract k-ends

The Fermi metric of $\mathbb{H}^3$ about a complete geodesic is

$$g = dr^2 + \sinh(r)^2 d\theta^2 + \cosh(r)^2 dt^2.$$

Given $f : mS^1 \times [0, \infty[\to \mathbb{R}$, define $e_f : mS^1 \times [0, \infty[\to \mathbb{H}^3$ by

$$e_f(\theta, t) := (f(\theta, t), \theta, t).$$

We call e_f a k-**end** of **order** m whenever it has constant extrinsic curvature equal to k and

$$\lim_{t \to +\infty} f(\theta, t) = 0.$$

We call f and **abstract** k-**end** of **order** m whenever this holds.

Every abstract k-end satisfies

$$f(\theta, t) = Ae^{-\sqrt{1-k}t} + \mathrm{o}(e^{-\sqrt{1-k}t}).$$

The first structure theorem

Theorem A - Let (S, e) be a finite-type k-surface. There exists a compact subset $K \subseteq S$ such that $(S \setminus K, e)$ is a disjoint union of finitely many finite-order k-ends.

The horizon map

The **horizon map** $h : \mathrm{U}\mathbb{H}^3 \to \partial_\infty \mathbb{H}^3 = \hat{\mathbb{C}}$ is defined by

$$h(u) := \lim_{t \to +\infty} \gamma_u(t),$$

where $\gamma_u : \mathbb{R} \to \mathbb{H}^3$ is the unique unit speed geodesic such that $\dot{\gamma}_u(0) = u$.

Let (S, e) be a k-surface. Let $N_e : S \to \mathrm{U}\mathbb{H}^3$ be its unit normal vector field. Define

$$\phi_e(x) := h \circ N_e.$$

This is a local homeomorphism and S carries a unique holomorphic structure $\phi^* \hat{\mathbb{C}}$ that makes ϕ_e holomorphic.

The second structure theorem

Theorem B - Let (S, e) be a finite-type k-surface. $S = \overline{S} \setminus P$, where $\overline{S}$ is a closed Riemann surface and $P \subseteq \overline{S}$ is finite. Moreover

(1) ϕ_e extends to a holomorphic function $\overline{\phi}_e : \overline{S} \to \hat{\mathbb{C}}$, the ramification points of $\overline{\phi}_e$ are elements of P and, for all $p \in P$, its ramification order at p is equal to $(m_p - 1)$, where m_p is the winding order of the end at p; and

(2) e extends to a continuous function $\overline{e} : \overline{S} \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ such that, for all $p \in P$, $\hat{e}(p) = \hat{\phi}_e(p)$.

A parametrization theorem

A **pointed ramified covering** of $\hat{\mathbb{C}}$ is a triple $(\overline{S}, P, \phi)$ where $\overline{S}$ is a closed Riemann surface, $P \subseteq \overline{S}$ is a finite subset, and $\phi : \overline{S} \to \hat{\mathbb{C}}$ is a holomorphic function with ramification points in P.

$\mathcal{R}$ is the space of pointed ramified coverings of $\hat{\mathbb{C}}$.

Theorem C - For all $k \in]0, 1[$ and for all $(\overline{S}, P, \phi) \in \mathcal{R}$, there exists a unique finite-type k-surface $e := e_\phi : \overline{S} \setminus P \to \mathbb{H}^3$ such that $\phi_e = \phi$.

The space of k-surfaces

The space of finite-type k-surfaces identifies with $\mathcal{R}$.

$\mathcal{R}$ is stratified by smooth complex manifolds locally parametrized by the images $(\phi(p_1), ..., \phi(p_m))$ of the generalized ramification points $P := (p_1, ..., p_m)$.

For each i, we call $z_i := \phi(p_i)$ the i'th **extremity** of e_ϕ.

Preferred geodesics

Every k-end is asymptotic to a unit-speed geodesic.

All unit-speed geodesics with the same end-point are asymptotic to one-another (up to translation).

However, k-ends have preferred geodesics.

Steiner curvature centroids

Consider first a convex subset $K \subseteq \mathbb{R}^2$ with support function $f : S^1 \to \mathbb{R}$.

Its **Steiner curvature centroid** is the first Fourier mode of f.

For triangles, it is the centre of mass obtained by placing weights at each vertex equal to the angle at that vertex.

This provides a concept of centre which is equivariant with respect to Minkowski addition.

This concept extends to locally strictly convex curves.

Steiner curvature centroids and k-ends

$\gamma : \mathbb{R} \to \mathbb{H}^3$ is a unit-speed geodesic.

$e : mS^1 \times [0, \infty[\to \mathbb{H}^3$ is a k-end about γ.

For all t, H_t is the unique horosphere centred on $\gamma(+\infty)$ passing through $\gamma(t)$.

For sufficiently large t, $e \pitchfork H_t$.

Let $c_{e,\gamma}(t)$ be the Steiner curvature centroid of this curve.

The third structure theorem

Theorem D - Let $e : mS^1 \times [0, \infty[\to \mathbb{H}^3$ be a k-end. There exists a unique unit-speed geodesic $\gamma : \mathbb{R} \to \mathbb{H}^3$ such that

$$d(\gamma(t), c_{e,\gamma}(t)) = \mathrm{O}(e^{-\sqrt{4-3k}t}).$$

We call this geodesic the **Steiner geodesic** of the k-end.

The extremity z of this end is an end-point of the Steiner geodesic.

We call the other end-point w of the Steiner geodesic the **Steiner point** of end.

A balancing formula

(S, e) is a finite-type k-surface. P is its set of ends.

For each $p \in P$, m_p is the winding order, z_p the extremity and w_p the Steiner point of the end at p.

For each $p \in P$, the **Steiner vector** of the p'th end is defined by

$$\zeta_p := \frac{1}{w_p - z_p}.$$

Theorem D - The Steiner vectors satisfy

$$\sum_{p \in P} m_p \zeta_p = 0.$$

Möbius invariance yields two other relations.

Lagrangian immersions

$(\overline{S}, P, \phi)$ is a ramified cover of $\hat{\mathbb{C}}$.

$\mathcal{R}_0$ is the stratum of $\mathcal{R}$ containing $(\overline{S}, P, \phi)$.

ω_0 is the symplectic form defined over $\hat{\mathbb{C}} \times \hat{\mathbb{C}} \setminus \Delta$ by

$$\omega_0 := \mathrm{Re}\left(\frac{1}{(z-w)^2} dz \wedge dw\right)$$

Theorem E - The function $F := (z_1, w_1, ..., z_N, w_N)$ is a lagrangian immersion from $\mathcal{R}_0$ into $(\hat{\mathbb{C}} \times \hat{\mathbb{C}} \setminus \Delta)^N$ furnished with the symplectic form

$$\omega := (m_1\omega_0) \oplus ... \oplus (m_N\omega_N).$$

Darboux coordinates

In the upper half-space model,

$$\mathrm{U}\mathbb{H}^3 = \left\{(x, y, z, u, v, w) \mid z > 0,\; u^2 + v^2 + w^2 = z^2\right\},$$

with contact form

$$\alpha := \frac{1}{z^2}(udx + vdy + wdz).$$

Darboux coordinates about a complete geodesic are

$$\Phi(x, y, u, v, t) := e^y \Bigg(t\cos(x) - u\sin(x), t\sin(x) + u\cos(x), 1, \frac{-\cos(x)}{\sqrt{1 + (t + v)^2}}, \frac{-\sin(x)}{\sqrt{1 + (t + v)^2}}, \frac{(t + v)}{\sqrt{1 + (t + v)^2}} \Bigg).$$

Abstract k-ends and Darboux coordinates

For $f : mS^1 \times [0, \infty[\to \mathbb{R}$, $\hat{f} : mS^1 \times [0, \infty[\to \mathbb{R}$ is

$$\hat{f}(\theta, t) := \Phi(\theta, t, f_\theta(\theta, t), f_t(\theta_t), t).$$

f is an abstract k-end whenever

$$kf_{\theta\theta} + f_{tt} - (1 - k)f = F(f, Df, D^2 f),$$

where F is analytic, vanishing up to order 2 at $(0, 0, 0)$.

Separation of variables

If $f : S^1 \times [0, \infty[\to \mathbb{R}$ is bounded and solves

$$f_{\theta\theta} + f_{tt} - c^2 f = 0,$$

then

$$f = \sum_{m \in \mathbb{Z}} a_m e^{im\theta} e^{-\sqrt{c^2+m^2}t}$$

Nonlinear separation of variables

If $f : S^1 \times [0, \infty[\to \mathbb{R}$ is bounded and solves

$$f_{\theta\theta} + f_{tt} - c^2 f = F(f, Df, D^2 f),$$

where F is smooth and vanishes up to order 2 at $(0, 0, 0)$.

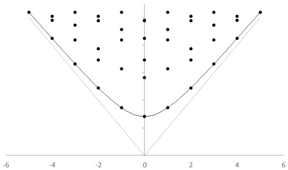
Then

$$f \sim \sum_{(\lambda,\mu)\in\mathcal{M}} a_{\lambda,\mu} e^{-i\lambda\theta} e^{\mu t}$$

where $\mathcal{M}$ is the additive sub-semigroup of $\mathbb{R} \times \mathbb{R}$ generated by

$$\mathcal{M}_0 := \left\{(m, \sqrt{m^2 + c^2}) \mid m \in \mathbb{Z}\right\}.$$

Structure of $\mathcal{M}$



After rescaling, the points $(0, \sqrt{1-k})$ and $(\pm 1, 1)$ are elements of $\mathcal{M}$.

The coefficient $a_{0,\sqrt{1-k}}$ is the "radius" of the cusp.

The coefficients $a_{\pm 1,1}$ yield the Steiner geodesic.

Thankyou!