On eternal forced mean curvature flows of tori in perturbations of the unit sphere

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Mean curvature flows

$M := M^m$, $N := N^{m+1}$ are smooth manifolds.

$I \subseteq \mathbb{R}$ is an open interval.

A **flow** is a smooth function $e : M \times I \to N$ such that $e_t := e(\cdot, t)$ is an immersion for all t.

When N is riemannian, the flow is a **mean curvature flow** (MCF) whenever

$$\left\langle \frac{\partial e_t}{\partial t}, N_t \right\rangle = -H_t \ \forall t,$$

where N_t and H_t are the unit normal vector field and the mean curvature of e_t.

Reparametrisations

A **reparametrisation** is a smooth function $\phi : M \times I \to M$ such that $\phi_t := \phi(\cdot, t)$ is a diffeomorphism for all t.

The MCF equation is reparametrisation invariant.

It is therefore highly degenerate.

When studying perturbations, the guage is fixed by considering only normal perturbations.

The MCF equation is then a quasi-linear parabolic equation near every flow.

Ancient, immortal and eternal flows

MCFs are be classified by their time domains.

Type	Interval
Ancient	$]-\infty, a[$
Immortal	$]a, \infty[$
Eternal	$\mathbb{R}$

The MCF equation is similar to a reaction-diffusion equation.

$\rightarrow$ Parabolicity regularises.

$\rightarrow$ Nonlinearity generates finite-time singularities.

$\Rightarrow$ Restrictions on the time-domain impose restrictions on solutions.

Hypoelliptic operators

The linearised MCF Operator is

$$L_\phi u := \left(\partial_t - \Delta - \phi\right) u,$$

where $\phi : M \times \mathbb{R} \to \mathbb{R}$ is smooth and bounded to all orders of differentiation.

For generic ϕ, L_ϕ is Fredholm.

Ker(L_ϕ) is finite-dimensional.

That is, it is small: eternal MCFs should be few and fairly rigid.

Known results - the convex case

MCF theory is poorly developed without some convexity hypothesis.

Theorem, Hamilton (1995)

Let e be an eternal MCF of locally strictly convex hypersurfaces in $\mathbb{R}^{m+1}$. If its mean curvature attains a maximum at some point of space-time, then e is a translating soliton.

Conjecture, White (2003)

Every eternal, convex, non-flat MCF in $\mathbb{R}^{m+1}$ is a translating soliton.

Main result I - Eternal MCFs

$\mathbb{T} := \mathbb{S}^1 \times \mathbb{S}^1$ is the standard torus.

$\mathbb{S}^3 \subseteq \mathbb{R}^4$ is the standard unit sphere with metric g_1.

Theorem, Magaño-S. (2020)

There exists a generic subset $\mathcal{U}$ of $C^\infty(\mathbb{S}^3)$ with the property that, for all $u \in U$, there exists $\epsilon > 0$ such that, for all $0 < |s| < \epsilon$, there exists a non-trivial eternal MCF $e : \mathbb{T} \times \mathbb{R} \to (\mathbb{S}^3, e^{2su} g_1)$.

Some definitions of function spaces

$\hat{\mathcal{E}} := \hat{\mathcal{E}}(M, N)$ is the space of smooth embeddings of M into N.

$\mathcal{E} := \mathcal{E}(M, N)$ is its quotient by Diff(M).

$\mathcal{E}$ identifies with $\mathcal{E}'$, the space of subsets of N which are smooth submanifolds diffeomorphic to M.

$\mathcal{E}$ is a smooth manifold (smooth-tame Frechet/weakly smooth).

$\mathcal{A}_g : \mathcal{E} \to \mathbb{R}$ is the area functional of the metric g.

Eternal MCFs are complete gradient flows of $\mathcal{A}_g$.

They contribute to the Morse complex of $(\mathcal{E}, \mathcal{A}_g)$.

Clifford Tori

The standard **Clifford torus**

$$T := \left\{ (x_1, x_2, y_1, y_2) \mid x_1^2 + x_2^2 = y_1^2 + y_2^2 = \frac{1}{2} \right\} \subseteq \mathbb{S}^3.$$

A **Clifford torus** is any image of T of $\mathrm{O}(4)$.

$\mathrm{CL} \subseteq \mathcal{E}$ is a smooth embedded submanifold.

$\mathrm{CL} \cong \mathrm{Gr}(2,4) \cong \mathbb{RP}^2 \times \mathbb{RP}^2$.

Main Result II - Morse Theory

$\mathrm{I} : C^\infty(\mathbb{S}^3) \to C^\infty(\mathrm{CL})$ is defined by

$$\mathrm{I}[u](T) := \int_T u\mathrm{dA}.$$

Theorem, Magaño-S. (2020)

There exists a generic subset $\mathcal{U}$ of $C^\infty(\mathbb{S}^3)$ with the property that, for all $u \in U$, there exists $\epsilon > 0$ such that, for all $0 < |s| < \epsilon$, the Morse complex of $(\mathrm{CL}, \mathrm{I}[u])$ is isomorphic to a subcomplex of the Morse complex of $(\mathcal{E}, \mathcal{A}_{e^{2su}g})$.

Perturbation theory of minimal hypersurfaces

$M \subseteq N$ is minimal.

$\mathcal{G}$ is the space of riemannian metrics over N.

For all g, for all f, $\mathcal{H}[g, f]$ is the mean curvature with respect to g of the normal graph of f over M.

$$\mathcal{H}[g, f] = Jf + \mathrm{O}(f^2) + \mathrm{O}(f(g - g_0)),$$

where J is the **Jacobi operator**.

When J is invertible, the Implicit Function Theorem yields minimal surfaces for nearby metrics.

For the Clifford Torus T, $J = (\Delta - 2)$.

Bundles

CL smoothly embeds in $\mathcal{E}$.

$T\mathcal{E}|_{\mathrm{CL}}$ identifies with the bundle $\mathcal{F}$CL whose fibre at T is $C^\infty(T)$.

TCL identifies with the subbundle whose fibre at T is $\mathrm{Ker}(\Delta - 2)$.

NCL identifies with the subbundle whose fibre at T is $\mathrm{Ker}(\Delta - 2)^\perp$.

$\pi^\perp : \mathcal{F}\mathrm{CL} \to N\mathrm{CL}$ is the orthogonal projection.

White's Result

Theorem, White (1991)

There exists a neighbourhood $\mathcal{U}$ of 0 in $C^\infty(\mathbb{S}^3)$ and a smooth functional $\Phi : \mathcal{U} \to \Gamma(N\mathrm{CL})$ such that, for all $u \in \mathcal{U}$ and for all $T \in \mathrm{CL}$ the mean curvature with respect to the metric $e^{2u}g_1$ of the normal graph of $\mathcal{F}[u](T)$ over T is an element of $\mathrm{Ker}(\Delta - 2)$.

For all $u \in \mathcal{U}$, for all T, the mean curvature with respect to $e^{2u}g_2$ of the normal graph of $\mathcal{F}[u](T)$ over T is as small as possible.

Since $T\mathrm{CL}$ has non-trivial self-intersection number, there exists T such that $\mathcal{F}[u](T) = 0$.

This proves existence of minimal surfaces for nearby metrics.

The area and mean curvature functionals

For $u \in \mathcal{U}$ and $T \in \mathrm{CL}$,

$\rightarrow$ $\mathrm{a}[u](T)$ is the area with respect to $e^{2u}g_1$ of the normal graph of $\mathcal{F}[u](T)$ over T.

$\rightarrow$ $\mathrm{h}[u](T)$ is the mean curvature with respect to $e^{2u}g_1$ of this normal graph.

Lemma

$$\mathrm{a}[su](T) = 2\pi^2 + s\mathrm{I}[u](T) + \mathrm{O}(s^2), \text{ and}$$
$$\mathrm{h}[su](T) = s\nabla\mathrm{I}[u](T) + \mathrm{O}(s^2)$$

If $\mathrm{I}[u]$ is of Morse type, its critical points identify with zeroes of $\mathrm{h}[su](T)$ for all small s.

The Morse-Smale property

$f : M \to \mathbb{R}$ is of **Morse-Smale type** whenever it is of Morse type and every stable manifold is transverse to every unstable manifold.

$\gamma : \mathbb{R} \to M$ is a complete gradient flow of f. Consider the operator $L : \Gamma(\gamma^* TM) \to \Gamma(\gamma^* TM)$

$$L_\gamma \sigma := \frac{\partial \sigma}{\partial t} + \mathrm{Hess}(f)(\gamma(t))\sigma(t).$$

When f is of Morse type, L_γ is always Fredholm.

When f is of Morse-Smale type, L_γ is always surjective.

Constructing eternal mean curvature flows

The Morse-Smale property ensure that complete gradient flows of I[u] perturb to complete gradient flows of h[su].

Care is still necessary:

$\rightarrow$ Flows of h[su] are not mean curvature flows;

$\rightarrow$ Allowing for this yields a genuinely singular perturbation problem;

$\rightarrow$ However, there are no nasty surprises.

Radon transforms

G is a Lie group.

$H_1, H_2 \subseteq G$ are compact subgroups.

For each i, $X_i := G/H_i$ and $\pi_i : G \to X_i$ is the canonical projection.

For each i, define the pull-back and push-forward:

$$(\pi_i^* f)(g) := f([g]), \text{ and}$$
$$(\pi_{i,*} f)([g]) := \int_{H_i} f(gh).$$

The Radon transform $R : C^\infty(X_i) \to C^\infty(X_j)$ is

$$\mathrm{R}[f] := (\pi_{j,*}\pi_i^*)(f).$$

The sphere and the space of Clifford tori

$\mathbb{S}^3 = \mathrm{O}(4)/\mathrm{O}(3)$.

$\mathrm{CL} = \mathrm{O}(4)/\mathrm{O}(2) \times \mathrm{O}(2)$.

Up to a constant, I is the Radon transform.

Theorem, Magaño-S. (2020)

For generic $u \in C^\infty(\mathbb{S}^3)$, $\mathrm{I}[u]$ is of Morse-Smale type.

Approximating delta functions

I$[u]$ is of Morse type.

$e : \mathbb{R} \to$ CL is a complete gradient flow of I$[u]$.

$e : \mathbb{T} \times \mathbb{R} \to \mathbb{S}^3$ is a flow of Clifford tori (with normal guage).

We require, for certain δ-functions over $\mathbb{T} \times \mathbb{R}$, $v \in C^\infty(\mathbb{S}^3)$ such that

$$\frac{\partial}{\partial t}(v \circ e)$$

approximates this function.

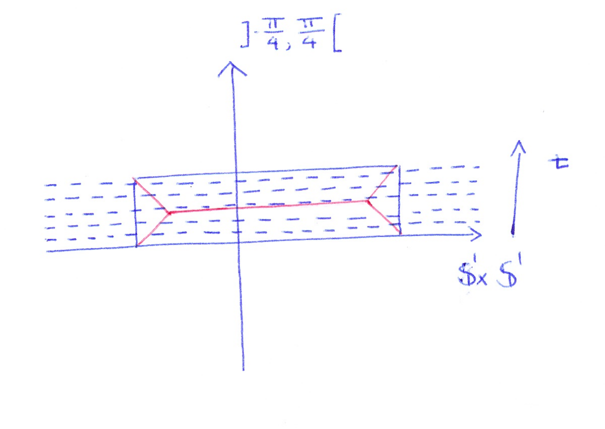
Fermi coordinates of the sphere

The spherical metric over $\mathbb{S}^1 \times \mathbb{S}^1 \times]-\pi/4, \pi/4[$ is

$$g := \sin^2(r + \pi/4) d\theta^2 + \sin^2(r - \pi/4) d\phi^2 + dr^2.$$

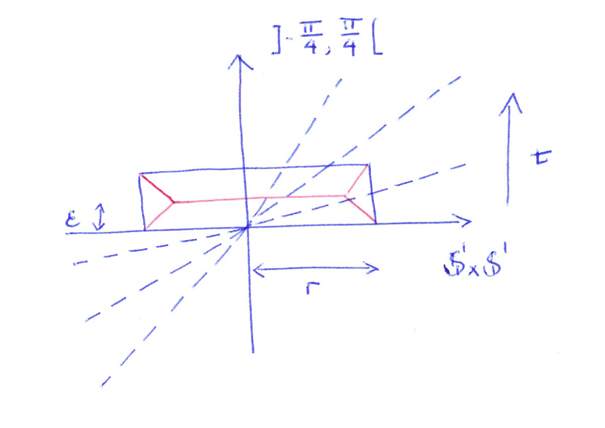
We construct Lipschitz bump functions with small support over this domain.

Constructing the bump function - Part I



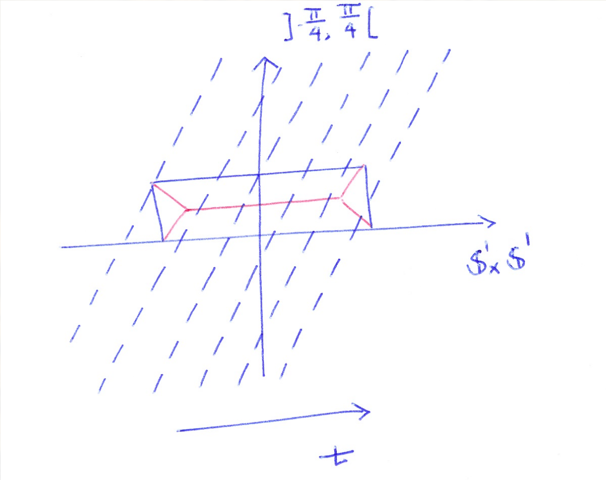
Near $\partial_t e \neq 0$, the contributions cancel.

Constructing the bump function - Part II



For non-cancelling contributions, choose $\partial_t e = 0$.

Constructing the bump function - Part III



For other tori in the flow, the contributions cancel.

Constructing the bump function - Part IV

The mass is roughly

$$M := \int_0^{\frac{\epsilon}{r}} \frac{2r^2h}{\epsilon} dt + \int_{\frac{\epsilon}{r}}^{\frac{2\epsilon}{r}} \frac{2rh}{\epsilon}\left(r - \frac{2\epsilon}{t}\right) dt$$
$$\geq 2rh.$$

Setting $rh = 1$ and $\epsilon = r^3$ yields, after much work, the desired function.

Morse-Smale functions and the Radon transform

$\mathbb{S}^3 = \mathrm{O}(4)/\mathrm{O}(3)$.

$\mathrm{CL} = \mathrm{O}(4)/\mathrm{O}(2) \times \mathrm{O}(2)$.

Theorem, Magaño-S. (2020)

For generic $u \in C^\infty(\mathbb{S}^3)$, $\mathrm{I}[u]$ is of Morse-Smale type.

Thankyou!