Groups of Hölder diffeomorphisms.

1 - Overview. Let X be a smooth, compact manifold. For all $k \in \mathbb{N}$ and for all $\alpha \in [0,1]$, let $\mathrm{Diff}^{k,\alpha}(X)$ denote the space of homeomorphisms of X which are of Hölder class (k,α). Trivially, this is a group except when $k+\alpha \in]0,1[$. However, for almost all values of k and α, it is not a topological group. Indeed, it is straightforward to show that the operations of composition and inversion are only continuous when $k=\alpha=0$. The purpose of this note is to show that continuity of these operations is recovered when $\mathrm{Diff}^{k,\alpha}(X)$ is replaced by the space $\mathrm{diff}^{k,\alpha}(X)$, defined to be the closure of $\mathrm{Diff}^{\infty}(X)$ in $\mathrm{Diff}^{k,\alpha}(X)$.

2 - Definitions. We recall the notation and terminology of Hölder norms and seminorms. Consider first two metric spaces X and Y. For all $f: X \to Y$ and for all $\alpha \in [0,1]$, the *Hölder seminorm* of f of order α is defined by

$$[f]_\alpha := \mathop{\mathrm{Sup}}_{x \neq y} \frac{d(f(x),f(y))}{d(x,y)^\alpha}. \tag{1}$$

In particular, $[f]_0$ is the *total variation* of f and $[f]_1$ is its *Lipschitz seminorm*. The following log-concavity property will prove useful.

Lemma 2.1

For all $f: X \to Y$, and for all $\alpha, \beta, t \in [0,1]$,

$$[f]_{t\alpha+(1-t)\beta} \leq [f]_\alpha^t [f]_\beta^{(1-t)}. \tag{2}$$

Proof: Indeed,

$$\begin{aligned} [f]_{t\alpha+(1-t)\beta} &= \mathop{\mathrm{Sup}}_{x\neq y} \frac{d(f(x),f(y))}{d(x,y)^{(t\alpha+(1-t)\beta)}} \\ &= \mathop{\mathrm{Sup}}_{x\neq y} \left(\frac{d(f(x),f(y))}{d(x,y)^\alpha}\right)^t \left(\frac{d(f(x),f(y))}{d(x,y)^\beta}\right)^{(1-t)} \\ &\leq [f]_\alpha^t [f]_\beta^{(1-t)}, \end{aligned}$$

as desired. □

Suppose now that Y is a normed vector space. For all $f: X \to Y$, the *uniform norm* of f is defined by

$$\|f\|_{C^0} := \mathop{\mathrm{Sup}}_{x} \|f(x)\|. \tag{3}$$

In particular, it is trivially related to the total variation of f by

$$[f]_0 \leq 2\|f\|_{C^0}.$$

For all $f: X \to Y$ and for all $\alpha \in [0,1]$, the *Hölder norm* of f of order $(0,\alpha)$ is defined by

$$\|f\|_{C^{0,\alpha}} := \|f\|_{C^0} + [f]_\alpha. \tag{4}$$

Suppose finally that X is an open subset of some normed vector space. The concept of derivatives of functions from X into Y can then be introduced. For all $k \in \mathbb{N}$, for all $\alpha \in [0,1]$, and for every k-times differentiable function $f : X \to Y$, the *Hölder norm* of f of order (k,α) is defined by

$$\|f\|_{C^{k,\alpha}} := \sum_{m=0}^{k} \|D^m f\|_{C^0} + [D^k f]_\alpha. \tag{5}$$

In particular, the Hölder norms satisfy the following inductive formula

$$\|f\|_{C^{k+1,\alpha}} = \|f\|_{C^0} + \|Df\|_{C^{k,\alpha}}.$$

For all (k,α), the *Hölder space* of order (k,α), denoted by $C^{k,\alpha}(X,Y)$, is defined to be the space of all k-times differentiable functions $f : X \to Y$ such that $\|f\|_{C^{k,\alpha}} < \infty$. It is straightforward to show that this space is non-separable and that $C^\infty(X,Y)$ is not a dense subset even when X is compact. For this reason, for all (k,α), the *little Hölder space* of order (k,α), denoted by $c^{k,\alpha}(X,Y)$, is defined to be the closure of $C^\infty(X,Y)$ in $C^{k,\alpha}(X,Y)$. In the special case where $\alpha = 1$, we have, for all k,

$$c^{k,1}(X,Y) = C^{k+1}(X,Y).$$

3 - Multilinear maps. In this section, E_1,...,E_m and F will be normed vector spaces, and $\mu : E_1 \oplus ... \oplus E_m \to F$ will be a bounded, multilinear map. We first suppose that X is a metric space.

Lemma 3.1

The map

$$C^0(X,E_1) \oplus ... \oplus C^0(X,E_m) \to C^0(X,F); (f_1,...,f_m) \mapsto \mu(f_1,...,f_m)$$

is a continuous, multilinear map of norm bounded by $\|\mu\|$.

Proof: Indeed, for all $f_1,...,f_m$,

$$\begin{aligned}\|\mu(f_1,...,f_m)\|_{C^0} &= \mathop{\mathrm{Sup}}_{x\in X} \|\mu(f_1,...,f_m)(x)\| \\ &= \mathop{\mathrm{Sup}}_{x\in X} \|\mu(f_1(x),...,f_m(x))\| \\ &\leq \mathop{\mathrm{Sup}}_{x\in X} \|\mu\|\|f_1(x)\|...\|f_m(x)\| \\ &\leq \|\mu\|\|f_1\|_{C^0}...\|f_m\|_{C^0},\end{aligned}$$

and the result follows. □

Lemma 3.2

For all $\alpha \in [0,1]$, the map

$$C^{0,\alpha}(X,E_1) \oplus ... \oplus C^{0,\alpha}(X,E_m) \to C^{0,\alpha}(X,F); (f_1,...,f_m) \mapsto \mu(f_1,...,f_m)$$

is a continuous, multilinear map of norm bounded by $\|\mu\|$.

Proof: It suffices to consider the case where $m=2$. For all f_1, f_2, and for all $\alpha \in [0,1]$,

$$\begin{aligned}
[\mu(f_1,f_2)]_\alpha &= \operatorname*{Sup}_{x\neq y} \frac{\|\mu(f_1,f_2)(x) - \mu(f_1,f_2)(y)\|}{d(x,y)^\alpha} \\
&= \operatorname*{Sup}_{x\neq y} \frac{\|\mu(f_1(x),f_2(x)) - \mu(f_1(y),f_2(y))\|}{d(x,y)^\alpha} \\
&\leq \operatorname*{Sup}_{x\neq y} \frac{\|\mu(f_1(x),f_2(x)) - \mu(f_1(y),f_2(x))\|}{d(x,y)^\alpha} \\
&\qquad + \operatorname*{Sup}_{x\neq y} \frac{\|\mu(f_1(y),f_2(x)) - \mu(f_1(y),f_2(y))\|}{d(x,y)^\alpha} \\
&\leq \|\mu\|[f_1]_\alpha\|f_2\|_{C^0} + \|\mu\|\|f_1\|_{C^0}[f_2]_\alpha,
\end{aligned}$$

and since

$$\|\mu(f_1,f_2)\|_{C^0} \leq \|\mu\|\|f_1\|_{C^0}\|f_2\|_{C^0},$$

it follows that

$$\|\mu(f_1,f_2)\|_{C^{0,\alpha}} \leq \|\mu\|\|f_1\|_{C^{0,\alpha}}\|f_2\|_{C^{0,\alpha}},$$

as desired. □

Suppose now that X is an open subset of a normed vector space.

Lemma 3.3

For all $k \in \mathbb{N}$ and for all $\alpha \in [0,1]$, the map

$$C^{k,\alpha}(X,E_1) \oplus ... \oplus C^{k,\alpha}(X,E_m) \to C^{k,\alpha}(X,F); (f_1,...,f_m) \mapsto \mu(f_1,...,f_m)$$

is a continuous, multilinear map of norm bounded by $m^k\|\mu\|$.

Proof: It suffices to consider the case where $m=2$. We prove this result by induction on k. The case where $k=0$ follows from Lemma 3.2. Denote by E the normed vector space in which X is contained and define the continuous bilinear maps $\mu_1 : \mathrm{Lin}(E,E_1) \oplus E_2 \to \mathrm{Lin}(E,F)$ and $\mu_2 : E_1 \oplus \mathrm{Lin}(E,E_2) \to \mathrm{Lin}(E,F)$ by

$$\begin{aligned}
&\mu_1(A,V)(U) := \mu(A(U),V), \text{ and} \\
&\mu_2(U,A)(V) := \mu(U,A(V)).
\end{aligned}$$

Consider now $f_1 \in C^{k+1,\alpha}(X, E_1)$ and $f_2 \in C^{k+1,\alpha}(X, E_2)$. By the chain rule,

$$D\mu(f_1, f_2) = \mu_1(Df_1, f_2) + \mu_2(f_1, Df_2),$$

so that, by the inductive hypothesis,

$$\|D\mu(f_1, f_2)\|_{C^{k,\alpha}} \leq 2^k \|\mu\| \left(\|Df_1\|_{C^{k,\alpha}} \|f_2\|_{C^{k,\alpha}} + \|f_1\|_{C^{k,\alpha}} \|Df_2\|_{C^{k,\alpha}}\right).$$

However, since

$$\|\mu(f_1, f_2)\|_{C^0} \leq \|\mu\| \|f_1\|_{C^0} \|f_2\|_{C^0},$$

it follows that

$$\|\mu(f_1, f_2)\|_{C^{k+1,\alpha}} \leq 2^{k+1} \|\mu\| \|f_1\|_{C^{k+1,\alpha}} \|f_2\|_{C^{k+1,\alpha}},$$

as desired. □

4 - Composition. In this section, X will be a metric space. We suppose first that Y and Z are also metric spaces and that Y is locally compact.

Lemma 4.1

The composition map

$$C^0(X, Y) \times C^0(Y, Z) \to C^0(X, Z); (f, g) \mapsto g \circ f$$

is continuous.

Proof: We prove this result using the compact-open topology. Consider an element (f, g) of $C^0(X, Y) \times C^0(Y, Z)$. Let U be an open subset of Z and let K be a compact subset of X such that $(g \circ f)(K) \subseteq U$. In particular, $f(K) \subseteq g^{-1}(U)$. Since $f(K)$ is compact and since Y is locally compact, there exists a relatively compact, open subset V of Y such that $K \subseteq V$ and $g(\overline{V}) \subseteq U$. Define now the neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of f and g respectively by

$$\mathcal{U} := \left\{f' \in C^0(X, Y) \mid f(K) \subseteq V\right\} \text{ and}$$
$$\mathcal{V} := \left\{g' \in C^0(X, Y) \mid g(\overline{V}) \subseteq U\right\}.$$

For all $(f', g') \in \mathcal{U} \times \mathcal{V}$, $(g' \circ f')(K) \subseteq U$, and the result follows. □

We henceforth suppose that Y and Z are subsets of normed vector spaces.

Lemma 4.2

For all $\alpha, \beta \in [0, 1]$, and for all $(f, g) \in C^{0,\alpha}(X, Y) \times C^{0,\beta}(Y, Z)$,

$$[g \circ f]_{\alpha\beta} \leq [g]_\beta [f]_\alpha^\beta. \tag{6}$$

In particular,

$$\|g \circ f\|_{C^{0,\alpha\beta}} \leq \|g\|_{C^{0,\beta}} \left(1 + [f]_\alpha^\beta\right).$$

Remark: It follows that pre-composition by an element of $C^{0,\alpha}(X, Y)$ defines a bounded linear map from $C^{0,\beta}(Y, Z)$ to $C^{0,\alpha\beta}(X, Z)$.

Proof: Indeed,

$$[g\circ f]_\alpha = \mathop{\mathrm{Sup}}_{x\neq y}\frac{\|(g\circ f)(x)-(g\circ f)(y)\|}{d(x,y)^{\alpha\beta}}$$
$$\leq [g]_\beta \mathop{\mathrm{Sup}}_{x\neq y}\frac{d(f(x),f(y))^\beta}{d(x,y)^{\alpha\beta}}$$
$$= [g]_\beta [f]_\alpha^\beta,$$

as desired. □

Lemma 4.3

For all $\alpha,\beta,\gamma\in[0,1]$ such that $\gamma<\alpha\beta$, the composition map

$$C^{0,\alpha}(X,Y)\times C^{0,\beta}(Y,Z)\to C^{0,\gamma}(X,Z);(f,g)\mapsto g\circ f$$

is continuous.

Proof: Indeed, consider a sequence (f_m,g_m) converging in $C^{0,\alpha}(X,Y)\times C^{0,\beta}(Y,Z)$ to (f_m,g_m). By Lemma 4.1,

$$\mathop{\mathrm{Lim}}_{m\to+\infty}\|(g_m\circ f_m)-(g_\infty\circ f_\infty)\|_{C^0}=0.$$

By (6), there exists $B>0$ such that, for all m,

$$[(g_m\circ f_m)-(g_\infty\circ f_\infty)]_{\alpha\beta}\leq [g_m\circ f_m]_{\alpha\beta}+[g_\infty\circ f_\infty]_{\alpha\beta}\leq B.$$

By (2) with $t:=\gamma/\alpha\beta$,

$$[(g_m\circ f_m)-(g_\infty\circ f_\infty)]_\gamma\leq B^t[(g_m\circ f_m)-(g_\infty\circ f_\infty)]_0^{(1-t)},$$

and since this tends to zero as m tends to infinity, the result follows. □

Lemma 4.4

For all $\alpha\in[0,1]$ and for all $\beta\in[0,1[$, the composition map

$$C^{0,\alpha}(X,Y)\times c^{0,\beta}(Y,Z)\to C^{0,\alpha\beta}(X,Z);(f,g)\mapsto g\circ f$$

is continuous.

Proof: Indeed, consider a sequence (f_m,g_m) converging in $C^{0,\alpha}(X,Y)\times c^{0,\beta}(Y,Z)$ to (f_∞,g_∞). For all m,

$$[(g_m\circ f_m)-(g_\infty\circ f_\infty)]_{\alpha\beta}\leq[(g_\infty\circ f_m)-(g_\infty\circ f_\infty)]_{\alpha\beta}+[(g_m-g_\infty)\circ f_m]_{\alpha\beta}.$$

By (6), the second term on the right hand side satisfies

$$[(g_m-g_\infty)\circ f_m]_{\alpha\beta}\leq[g_m-g_\infty]_\beta[f_m]_\alpha^\beta,$$

which tends to zero as m tends to infinity. Consider now the first term on the right hand side. Let $B > 0$ be such that, for all $m \in \mathbb{N} \cup \{\infty\}$,

$$[f_m]_\alpha \leq B.$$

Now choose $\epsilon > 0$ and choose $h \in C^\infty(Y, Z)$ such that

$$[h - g_\infty]_\beta \leq \epsilon/3B^\beta.$$

Using (6) again, we obtain

$$\begin{aligned}[(g_\infty \circ f_m) - (g_\infty \circ f_\infty)]_{\alpha\beta} &\leq [(h - g_\infty) \circ f_m]_{\alpha\beta} \\ &\qquad + [(h \circ f_m) - (h \circ f_\infty)]_{\alpha\beta} \\ &\qquad\qquad + [(h - g_\infty) \circ f_\infty]_{\alpha\beta} \\ &\leq [(h \circ f_m) - (h \circ f_\infty)]_{\alpha\beta} \\ &\qquad + [h - g_\infty]_\beta [f_m]_\alpha^\beta \\ &\qquad\qquad + [h - g_\infty]_\beta [f_\infty]_\alpha^\beta \\ &\leq [(h \circ f_m) - (h \circ f_\infty)]_{\alpha\beta} + 2\epsilon/3.\end{aligned}$$

Since $h \in C^{0,1}(Y, Z)$, it follows by Lemma 4.3 that, for sufficiently large m,

$$[(h \circ f_m) - (h \circ f_\infty)]_{\alpha\beta} \leq \epsilon/3,$$

so that

$$[(g_\infty \circ f_m) - (g_\infty \circ f_\infty)]_{\alpha\beta} \leq \epsilon.$$

Since ϵ may be chosen arbitrarily small, the first term on the right hand side also tends to zero as m tends to infinity, and this completes the proof. □

The case where $\beta = 1$ is treated separately. Although it is not strictly necessary for our purposes, we include it for completeness.

Lemma 4.5

If Y is convex and compact then, for all $\alpha \in [0, 1]$, the composition map

$$C^{0,\alpha}(X, Y) \times C^1(Y, Z) \to C^{0,\alpha}(X, Z); (f, g) \mapsto g \circ f$$

is continuous.

Proof: Let E and F denote the normed vector spaces containing Y and Z respectively. Suppose furthermore that F is complete, so that the integral of continuous curves in F is well defined. Now let (f_m, g_m) be a sequence converging in $C^{0,\alpha}(X, Y) \times C^1(Y, Z)$ to (f_∞, g_∞). Bearing in mind that Y is convex, for all $m \in \mathbb{N} \cup \{\infty\}$, we define $A_m : X \times X \to \mathrm{Lin}(E, F)$ by

$$A_m(x, y) := \int_0^1 Dg_m((1-t)f_m(x) + tf_m(y))dt.$$

It follows by compactness of Y that the sequence (A_m) converges uniformly to A_∞ as m tends to infinity. Now, for all m,

$$\begin{aligned}[g\circ f_m - g\circ f_\infty]_\alpha &= \mathop{\mathrm{Sup}}_{x\neq y}\frac{\|A_m(x,y)(f_m(y)-f_m(x)) - A_\infty(x,y)(f_\infty(y)-f_\infty(x))\|}{d(x,y)^\alpha}\\ &\leq \mathop{\mathrm{Sup}}_{x\neq y}\frac{\|A_m(x,y)(f_m(y)-f_m(x)-f_\infty(y)+f_\infty(x))\|}{d(x,y)^\alpha}\\ &\qquad + \mathop{\mathrm{Sup}}_{x\neq y}\frac{\|(A_m(x,y)-A_\infty(x,y))(f_\infty(y)-f_\infty(x))\|}{d(x,y)^\alpha}\\ &\leq \|A_m\|_{C^0}[f_m - f_\infty]_\alpha + \|A_m - A_\infty\|_{C^0}[f_\infty]_\alpha,\end{aligned}$$

and since this tends to zero as m tends to infinity, the result follows. □

Finally, we suppose that X is an open subset of a normed vector space.

Lemma 4.6

For all $k\geq 1$, and for all $\alpha\in[0,1]$, the composition map

$$C^{k,\alpha}(X,Y)\times c^{k,\alpha}(Y,Z)\to C^{k,\alpha}(X,Z); (f,g)\mapsto g\circ f$$

is continuous.

Proof: Since $C^{k,1}=C^{k+1}$, the case where $\alpha=1$ follows by a straightforward argument of elementary calculus. We therefore suppose that $\alpha<1$, and we prove this result by induction in k. Consider a sequence (f_m,g_m) converging to (f_∞,g_∞) in $C^{k,\alpha}(X,Y)\times c^{k,\alpha}(Y,Z)$. By Lemma 4.1, the sequence $(g_m\circ f_m)$ converges to $(g_\infty\circ f_\infty)$ in $C^0(X,Z)$. By the chain rule, for all $m\in\mathbb{N}\cup\{\infty\}$,

$$D(g_m\circ f_m) = (Dg_m\circ f_m)Df_m.$$

Denote by E and F the normed vector spaces containing X and Z respectively. If $k>1$, then it follows by the inductive hypothesis that the sequence $(Dg_m\circ f_m)$ converges to $(Dg_\infty\circ f_\infty)$ in $C^{k-1,\alpha}(X,\mathrm{Lin}(E,F))$. Otherwise, if $k=1$, then this property follows by Lemma 4.4. In each case, by Lemma 3.3, the sequence $(D(g_m\circ f_m))$ converges to $D(g_\infty\circ f_\infty)$ in $C^{0,\alpha}(X,\mathrm{Lin}(E,F))$, and we conclude that the sequence $(g_m\circ f_m)$ converges to $(g_\infty\circ f_\infty)$ in $C^{k,\alpha}(X,Z)$, as desired. □

Suppose now that X is a smooth, compact, embedded submanifold of some finite-dimensional vector space, and observe that the above results continue to hold in this case. Let $\mathrm{diff}^{k,\alpha}(X)$ denote the space of diffeomorphisms of X which are of type $c^{k,\alpha}$. Setting $Z=Y=X$, Lemma 4.6 immediately yields

Theorem 4.7

For all $k\geq 1$, and for all $\alpha\in[0,1]$, the composition map

$$\mathrm{diff}^{k,\alpha}(X)\times\mathrm{diff}^{k,\alpha}(X)\to\mathrm{diff}^{k,\alpha}(X); (f,g)\mapsto g\circ f$$

is continuous.

We conclude by proving continuity of the inversion map. First, we have

Lemma 4.8

The inversion map

$$\mathrm{homeo}(X) \to \mathrm{homeo}(X); \phi \to \phi^{-1}$$

is continuous.

Proof: Indeed, consider a sequence (ϕ_m) converging in $\mathrm{Homeo}(X)$ to ϕ_∞. Let U and K be respectively an open and a compact subset of X such that $\phi_\infty^{-1}(K) \subseteq U$. In particular $\phi_\infty(U^c) \subseteq K^c$ so that, since U^c is compact and K^c is open, for sufficiently large m, $\phi_m(U^c) \subseteq K^c$. It follows that, for sufficiently large m, $\phi_m^{-1}(K) \subseteq U$ and so (ϕ_m^{-1}) converges to ϕ_∞^{-1} in the compact-open topology, as desired. $\square$

Theorem 4.9

For all $k \geq 1$ and for all $\alpha \in [0,1]$, the inversion map

$$\mathrm{diff}^{k,\alpha}(X) \to \mathrm{diff}^{k,\alpha}(X); \phi \mapsto \phi^{-1}$$

is continuous.

Proof: We prove this by induction on k. Consider a sequence (ϕ_m) converging in $\mathrm{diff}^{k,\alpha}(X)$ to ϕ_∞ and for all $m \in \mathbb{N} \cup \{\infty\}$ denote $\psi_m := \phi_m^{-1}$. By Lemma 4.8, (ψ_m) converges to ψ_∞ in the C^0 sense. By the chain rule, for all $m \in \mathbb{N} \cup \{\infty\}$,

$$D\psi_m = (D\phi_m)^{-1} \circ \psi_m. \tag{7}$$

We now claim that (ψ_m) converges to ψ_∞ in the $C^{k-1,\alpha}$ sense. Indeed, when $k > 1$, this follows by the inductive hypothesis. Otherwise, when $k = 1$, we first observe that (7) implies that $(D\psi_m)$ converges towards $D\psi_\infty$ in the C^0 sense. It follows that (ψ_m) converges to ψ_∞ in the C^1 sense, and therefore also in the $C^{0,\alpha}$ sense, as asserted. In each case, it follows by Lemma 4.6 that $(D\psi_m)$ converges towards $D\psi_\infty$ in the $C^{k-1,\alpha}$ sense, and so (ψ_m) converges towards ψ_∞ in the $C^{k,\alpha}$ sense, as desired. $\square$