1 - On the Cheeger-Gromov topology.

Given its rather technical definition, the Cheeger-Gromov topology can often appear quite daunting at first sight. In this appendix, we present elementary results which we believe are key to its understanding. As an application, we prove the Hausdorff property of this topology up to isometries that preserve the base points.

First, a formal observation. Technically, the space of all finite-dimensional manifolds is not actually a set. However, using partitions of unity, every such manifold is realised in a straightforward manner as an embedded submanifold of $L^2([0,1])$, and since the space of all such submanifolds *is* a set, it is nonetheless valid to speak of topologies over the space of finite-dimensional manifolds.

Lemma 0.1

Let X and Y be finite-dimensional manifolds. Let $(\Phi_m) : X \to Y$ be a sequence of homeomorphisms onto open subsets of Y converging in the compact-open topology to the map $\Phi_\infty : X \to Y$ which is also a homeomorphism onto an open subset of Y.

For every compact subset $K \subseteq Im(\Phi_\infty)$, there exists $M(K) \in \mathbb{N}$ such that, for $m \geq M(K)$, $K \subseteq (Im(\Phi_m))$. In particular, the sequence of restrictions $(\Phi_m^{-1}|_K)_{m\geq M(K)}$ converges in the compact-open topology to $\Phi_\infty^{-1}|_K$.

Proof: Let y be a point of K and denote $x := \Phi_\infty^{-1}(y)$. Let $\Psi : \mathbb{R}^d \to Y$ be a diffeomorphisms onto an open subset of Y such that $\Psi(0) = y$. For all $r > 0$, let B_r denote the image under the action of Ψ of the open, unit ball of radius r about 0 in $\mathbb{R}^d$, and let $\overline{B}_r$ denote its closure. Now let $r > 0$ be such that B_{2r} is contained in $\mathrm{Im}(\Phi_\infty)$, and denote $X_r := \Phi_\infty^{-1}(\overline{B}_r)$. Since X_r and ∂X_r are compact, for sufficiently large m, $\Phi_m(x) \in B_{r/2}$, $\Phi_m(X_r) \subseteq B_{3r/2}$ and $\Phi_m(\partial X_r) \subseteq B_{3r/2} \setminus B_{r/2}$. Since Φ_m is a homeomorphism, it follows that $B_{r/2} \subseteq \mathrm{Im}(\Phi_m)$. The first assertion now follows by compactness of K.

To prove the second assertion, consider a compact subset L of K and an open subset Ω of X such that $L \subseteq \Phi_\infty(\Omega)$. By the first assertion, with X now replaced by Ω, for sufficiently large m, $L \subseteq \Phi_m(\Omega)$. Convergence of the restrictions of the inverses in the compact open topology now follows, and this completes the proof. □

Corollary 0.2

Let X and Y be manifolds and let $(\Phi_m) : X \to Y$ be a sequence of smooth diffeomorphisms onto open subsets of Y converging in the C^∞_{loc} sense to the map $\Phi_\infty : X \to Y$ which is also a smooth diffeomorphism onto an open subset of Y.

For every relatively compact open subset Ω of $Im(\Phi_\infty)$, the sequence of restrictions $(\Phi_m^{-1}|_\Omega)_{m\geq M(\overline{\Omega})}$ converges in the C^∞ sense to $\Phi_\infty^{-1}|_\Omega$, where here $M(\overline{\Omega})$ is as in the statement of Lemma 0.1.

Proof: Indeed, this follows immediately from Lemma 0.1 and the chain rule. □

Lemma 0.3

Let X and Y be smooth manifolds. Let (g_m) and (h_m) be sequences of smooth riemannian metrics over X and Y respectively converging in the C^∞_{loc} sense to the smooth riemannian metrics g_∞ and h_∞ respectively. Let K be a compact subset of Y and let $(\Phi_m) : X \to K$

be a sequence of smooth maps such that, for all m, $\Phi_m^* h_m = g_m$. There exists a smooth map $\Phi_\infty : X \to K$ towards which (Φ_m) subconverges in the C^∞_{loc} sense.

Proof: First, for every point x in X, for all $m \in \mathbb{N} \cup \{\infty\}$ and for all $r > 0$, let $N_r(x, m)$ denote the geodesic neighbourhood of radius r about x in X with respect to the metric g_m, and let $B_r(x, m)$ denote the ball of radius r in $T_x X$ with respect to the same metric.

Let X_0 be a countable, dense subset of X. By compactness and a diagonal argument, there exists $\Phi_\infty : X_0 \to K$ towards which the sequence of restrictions $(\Phi_m|_K)$ subconverges. Let Ω be a relatively compact open subset of X. By compactness, there exists $r > 0$ such that, for all m, and for all $x \in K$, the injectivity radius of the metric g_m about the point x is at least r. Consider now a point $x \in X_0 \cap \Omega$. For all $m \in \mathbb{N} \cup \{\infty\}$, let $E_{m,g} := \mathrm{Exp}_{g_m, x}$ and $E_{m,h} := \mathrm{Exp}_{h_m, \Phi_m(x)}$ denote the respective exponential maps of g_m and h_m about the points x and $\Phi_m(x)$. Then, for all m, and for all $\xi \in B_r(x, m)$,

$$(\Phi_m \circ E_{m,g})(\xi) = (E_{m,h} \circ D\Phi_m)(\xi),$$

so that, for all $x' \in N_r(x, m)$,

$$\Phi_m(x') = (E_{m,h} \circ D\Phi_m \circ E_{m,g}^{-1})(x').$$

By Lemma 0.1, there exists $M > 0$ such that, for $m \geq M$, $N_{r/2}(x, \infty) \subseteq E_{m,g}(B_r(x, m))$. Furthermore, by Corollary , the sequence of restrictions $(E_{m,g}^{-1}|_{N_{r/2}(x,\infty)})_{m \geq M}$ converges in the C^∞ sense to $E_{\infty,g}^{-1}$. However, since the sequence $(D\Phi_m(x))$ is uniformly bounded, it subconverges to a limit, A, say. The sequence (Φ_m) therefore subconverges in the C^∞ sense over $N_{r/2}(x, \infty)$ to $E_{\infty,h} \circ A \circ E_{\infty,g}^{-1}$, and the result now follows by a diagonal argument. □

It is now straightforward to recover the Hausdorff property of the Cheeger-Gromov topology.

Theorem 0.4

Let (X_m, g_m, x_m) be a sequence of pointed, complete riemannian manifolds. If this sequence converges towards $(X_\infty, g_\infty, x_\infty)$ and $(X'_\infty, g'_\infty, x'_\infty)$ in the Cheeger-Gromov sense, then there exists an isometric diffeomorphism $\Phi : X_\infty \to X'_\infty$ such that $\Phi(p_\infty) = p'_\infty$.

Proof: Let (Φ_m) and (Φ'_m) be a sequence of convergence maps of (X_m, g_m, x_m) with respect to the limits $(X_\infty, g_\infty, x_\infty)$ and $(X'_\infty, g'_\infty, x'_\infty)$ respectively. Fix $r > 0$, and let V denote the open geodesic neighbourhood of radius r about x'_∞ in X'_∞. Let $M \in \mathbb{N}$ be such that, for all $m \geq M$, the function Φ'_m defines a smooth diffeomorphism of V into its image. Upon increasing M if necessary, we may suppose that $\Phi'_m(V)$ lies in the image of Φ_m, and that $((\Phi_m)^{-1} \circ \Phi'_m)(V)$ contains the geodesic ball of radius $r/2$ about x_∞ in X_∞, which we denote by U.

Now, for any manifold Y, for any riemannian metric h over this manifold, and for any positive, integer k, let $\| \cdot \|_{C^k(Y,h)}$ denote the C^k norm of a section over Y with respect to the metric h. By hypothesis,

$$\operatorname*{Lim}_{m \to \infty} \|(\Phi'_m)^* g_m - g'_\infty\|_{C^k(V, g'_\infty)} = 0.$$

It follows that

$$\begin{aligned} &\operatorname*{Lim}_{m\to\infty} \|(\Phi'_m)^* g_m - g'_\infty\|_{C^k(V,(\Phi'_m)^* g_m)} &&= 0 \\ \Rightarrow\quad &\operatorname*{Lim}_{m\to\infty} \|g_m - (\Phi'_m)_* g'_\infty\|_{C^k(\Phi'_m(V),g_m)} &&= 0. \\ \Rightarrow\quad &\operatorname*{Lim}_{m\to\infty} \|(\Phi_m)^* g_m - (\Phi_m)^*(\Phi_m)_*^{-1} g'_\infty\|_{C^k(U,\Phi_m^* g_m)} &&= 0. \end{aligned}$$

However, by hypothesis again,

$$\operatorname*{Lim}_{m\to\infty} \|(\Phi_m)^* g_m - g_\infty\|_{C^k(U,g_\infty)} = 0, \tag{1}$$

so that

$$\operatorname*{Lim}_{m\to\infty} \|(\Phi_m)^* g_m - (\Phi_m)^*(\Phi_m)_*^{-1} g'_\infty\|_{C^k(U,g_m)} = 0. \tag{2}$$

By (1), (2) and the triangle inequality, the sequence of metrics $((\Phi'_m \circ (\Phi_m)^{-1})^* g'_\infty)_{m\geq M}$ converges in the C^∞ sense over U to g_∞.

By Lemma 0.3, $((\Phi'_m)^{-1} \circ \Phi_m)_{m\geq M}$ converges in the C^∞_{loc} sense over U to a smooth map $\Psi_\infty : U \to X'_\infty$ which is a diffeomorphism onto its image and which sends x_∞ to x'_∞. Upon applying a diagonal argument we may suppose furthermore that Ψ_∞ is defined over the whole of X_∞. Finally, $\Psi^*_\infty g'_\infty = g_\infty$. That is, Ψ_∞ is everywhere a local isometry, and since both X_∞ and X'_∞ are complete, it follows that Ψ_∞ is surjective. This completes the proof. □